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<p>The weighted least squares fit of a real tone with arbitrary amplitude, frequency, and phase, to a given set of real discrete data, is reduced to a one-dimensional maximization of a function of frequency only. This function is manipulated into a form that can be efficiently calculated by one FFT of a complex sequence that is related to the available real data and the arbitrary real weight sequence utilized. The decoupling of the complex FFT outputs, to yield the two functions that are necessary to conduct the coarse search in frequency, is accomplished in an extremely simple fashion. A refined interpolation procedure then fits a parabola in the region near the maximum and gives a fine-grained estimate of frequency.</p> <p>An explanation of the apparently anomalous behavior near zero and Nyquist</p>					
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frequencies is given, which shows that in the limit, a constant plus linear trend is being fitted to the discrete data. A program is presented for the complete procedure, including evaluation of the best frequency, amplitude, and phase of the fitted tone. The technique is applicable to short data records, without any approximations, and for arbitrary weight sequences.

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# **Weighted Least Squares Fit of a Real Tone to Discrete Data, by Means of an Efficient Fast Fourier Transform Search**

✓ Albert H. Nuttall  
Surface Ship Sonar Department



**Naval Underwater Systems Center**  
// Newport, Rhode Island / New London, Connecticut

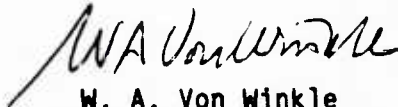
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## Preface

This research was conducted under NUSC Project No. A75205, Subproject No. ZR0000101, "Applications of Statistical Communication Theory to Acoustic Signal Processing," Principal Investigator Dr. Albert H. Nuttall (Code 3314), sponsored by the NUSC In-House Independent Research Program, Program Manager Mr. W. R. Hunt, Director of Navy Laboratories (SPAWAR 05).

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A handwritten signature in dark ink, appearing to read 'WA Von Winkle', is written over the printed name.

W. A. Von Winkle  
Associate Technical Director  
for Technology

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## LIST OF SYMBOLS

$N$	Number of data points
$x_k$	Data value at sample number $k$
$\Delta$	Sample increment
$E$	Weighted squared error, (7)
$w_k$	Weight at sample number $k$
$\omega$	Radian frequency of tone
$\alpha, \beta$	Coefficients of in-phase and quadrature components, (5)
FFT	Fast Fourier Transform
$a$	Normalized frequency $\omega\Delta$ , (6),(7)
$W$	Window for weights $\{w_k\}$ , (8)
$L$	Window for weighted data, (8)
sub $r, i$	Real and imaginary parts, respectively
$A_{mn}$	Auxiliary quantities, (9),(10)
$B(a)$	Maximizing function, (12),(13)
Re	Real part
$W_n$	$n$ -th moment of weights, (24)
$L_n, \tilde{L}_n$	$n$ -th moment of weighted data, (24),(31)
$\mu, \nu$	Constant and linear trend, (26)
$\tilde{a}$	Perturbation of $a$ about $\pi$ , (29)
$M$	Size of FFT, (53),(54)
$m$	Frequency index, (53)
$\mathcal{L}(m)$	FFT of weighted data, (54)
$d_j$	Auxiliary sequence, (56)
$\mathcal{D}(m)$	FFT of $\{d_j\}$ , (57)
$z_k$	Complex FFT input, (59)
$Z(m)$	Complex FFT output, (60)
$S, D$	Sum and difference variables, (63)
$X, Y$	Real and imaginary parts of $Z$ , (64)

WEIGHTED LEAST SQUARES FIT OF A REAL TONE TO DISCRETE DATA,  
BY MEANS OF AN EFFICIENT FAST FOURIER TRANSFORM SEARCH

INTRODUCTION

Estimation of the parameters of a tone with unknown amplitude, frequency, and/or phase has attracted considerable attention; see, for example, [1-9]. However, fitting data with a single pure complex tone leads to a simpler search problem than fitting with a real tone (as will be demonstrated in the next section). In particular, fitting with a complex tone was considered in [1-5, 7], while fitting with real tones has been the subject of [6, 8, 9]. However, the frequency of the tone was assumed known in [6, 8], whereas it had to be estimated in [9].

Here we will extend the results in several directions for the case of fitting real data with a real tone. First, arbitrary real weighting of the errors at each discrete instant are incorporated. Second, the function that must be searched for a maximum is manipulated into a form which requires that only two FFTs of two real sequences be conducted. Third, these two operations are combined into one FFT of a complex sequence, the outputs of which are decoupled in a very efficient manner, in order to yield the desired search function. Fourth, parabolic interpolation of the three outputs in the neighborhood of the search maximum is employed in order to give a refined estimate of the tone frequency. Finally, a minute search for the best tone frequency is conducted, the extent of which is left up to the user. The end result of this investigation is a program for conducting an efficient and fast fine-grained search for the determination of the unknown amplitude, frequency, and phase of the best-fitting real tone to a given set of discrete real data and subject to any error weighting of interest.

This procedure is applicable to arbitrary data record lengths. Also, no assumptions about the statistics of any additive noise, that may be present in the data record, are made. However, when the available data record is the result of a pure tone and additive zero-mean Gaussian noise, the procedure can be interpreted as maximum likelihood estimation [9].

## ERROR MINIMIZATION

Before we begin the detailed investigation of fitting a real tone to real data, we first consider the simpler problem of fitting a pure complex tone. This will serve as a comparison procedure and will back up the statement made in the Introduction.

## COMPLEX TONE

The discrete data available consist of  $N$  values  $\{x_k\}$ , taken at increment  $\Delta$ . If the data are complex and we fit the data with a pure complex tone, we must address the problem of minimizing the weighted squared error

$$E = \sum_k w_k |x_k - \alpha \exp(i\omega k\Delta)|^2, \quad (1)$$

where the summation on  $k$  is taken over all nonzero summands. Normally, the data  $\{x_k\}$  and real weights  $\{w_k\}$  will be taken to be nonzero over the range  $1 \leq k \leq N$ ; however, the presentation allows for any range of the variable  $k$ . The parameter  $\alpha$  in (1) is the complex amplitude, and  $\omega$  is the pure tone (radian) frequency, which is presumed real.

If we consider  $\omega$  given for the moment in (1), the best choice of  $\alpha$  to minimize error  $E$  is given by

$$\alpha_0 = \sum_k w_k x_k \exp(-i\omega k\Delta) / \sum_k w_k. \quad (2)$$

Substitution of this result for  $\alpha$  in (1) results in error

$$E(\omega) = \sum_k w_k |x_k|^2 - \left| \sum_k w_k x_k \exp(-i\omega k\Delta) \right|^2 / \sum_k w_k. \quad (3)$$

This error is minimized by choosing frequency  $\omega$  to maximize the quantity

$$\left| \sum_k w_k x_k \exp(-i\omega k\Delta) \right|^2, \quad (4)$$

which is the standard magnitude-squared Fourier transform of the weighted data. Thus, direct application of an FFT is a good procedure to apply to this problem and has been so employed in the past [5]. Since (4) has period  $2\pi/\Delta$  in  $\omega$ , there is no need to compute (4) except for the range  $-\pi \leq \omega\Delta \leq \pi$ .

#### REAL TONE

We now restrict consideration to the case of major interest here, namely, real data  $\{x_k\}$ , and attempt to fit it with samples of a pure real tone, that is,

$$\alpha \cos(\omega k\Delta) + \beta \sin(\omega k\Delta). \quad (5)$$

Here,  $\alpha$  and  $\beta$  are the real coefficients of the in-phase and quadrature components of the tone. If we let "normalized frequency"

$$a = \omega\Delta, \quad (6)$$

the weighted squared error to be minimized is

$$E = \sum_k w_k [x_k - \alpha \cos(ak) - \beta \sin(ak)]^2. \quad (7)$$

For later use, we define the two Fourier series:

$$W(u) = \sum_k w_k \exp(-iuk),$$

$$L(u) = \sum_k w_k x_k \exp(-iuk). \quad (8)$$

The first is the window associated with weights  $\{w_k\}$ , while the latter is the Fourier transform of the weighted data.

The variable  $a$  appearing in (7) will be called the "frequency" of the tone. If we consider frequency  $a$  given for the moment, setting the partial derivatives of error  $E$  with respect to  $\alpha$  and  $\beta$ , both equal to zero, results in the pair of simultaneous linear equations for their optimum values:

$$\begin{aligned} A_{11} \alpha_0 + A_{12} \beta_0 &= L_r(a) , \\ A_{12} \alpha_0 + A_{22} \beta_0 &= -L_i(a) . \end{aligned} \quad (9)$$

Here sub  $r$  and  $i$  denote real and imaginary parts, respectively. We also have the scale factors expressible in the forms

$$\begin{aligned} A_{11} &= \sum_k w_k \cos^2(ak) = \frac{1}{2} [W(0) + W_r(2a)] , \\ A_{22} &= \sum_k w_k \sin^2(ak) = \frac{1}{2} [W(0) - W_r(2a)] , \\ A_{12} &= \sum_k w_k \cos(ak) \sin(ak) = -\frac{1}{2} W_i(2a) , \end{aligned} \quad (10)$$

where we have made extensive use of (8). Solution of (9) yields for the tone coefficients,

$$\begin{aligned} \alpha_0 &= \frac{A_{22} L_r(a) + A_{12} L_i(a)}{A_{11} A_{22} - A_{12}^2} , \\ \beta_0 &= \frac{-A_{11} L_i(a) - A_{12} L_r(a)}{A_{11} A_{22} - A_{12}^2} . \end{aligned} \quad (11)$$

The use of (9)-(11) in error (7) now results in modified error

$$\begin{aligned}
E(a) &\equiv \sum_k w_k [x_k - \alpha_0 \cos(ak) - \beta_0 \sin(ak)]^2 = \\
&= \sum_k w_k [x_k - \alpha_0 \cos(ak) - \beta_0 \sin(ak)] x_k = \\
&= \sum_k w_k x_k^2 - \alpha_0 L_r(a) + \beta_0 L_i(a) = \\
&= \sum_k w_k x_k^2 - B(a) , \tag{12}
\end{aligned}$$

where we define real function

$$\begin{aligned}
B(a) &= \alpha_0 L_r(a) - \beta_0 L_i(a) = \\
&= \frac{A_{22}L_r^2(a) + 2A_{12}L_r(a)L_i(a) + A_{11}L_i^2(a)}{A_{11}A_{22} - A_{12}^2} . \tag{13}
\end{aligned}$$

This quantity, which must now be maximized by choice of  $a$ , was previously encountered in [9; (10)], but limited there to the case of equal weights  $\{w_k\}$ . We concentrate henceforth on function  $B(a)$ , aware that we can always return to error  $E(a)$  by means of (12).

MANIPULATIONS OF  $B(a)$ 

In this section, we derive alternative forms, properties, and interpretations of the function  $B(a)$ . The weighted squared error is directly related to  $B(a)$  by means of (12).

ALTERNATIVE FORM FOR  $B(a)$ 

A more useful and compact form for  $B(a)$  in (13) is possible. Reference to (10) reveals that the denominator of (13) is simply

$$\frac{1}{4} [W^2(0) - |W^2(2a)|] . \quad (14)$$

Similarly, use of (10) allows development of the numerator of (13) according to

$$\begin{aligned} & \frac{1}{2} [W(0) - W_r(2a)] L_r^2(a) - W_i(2a) L_r(a) L_i(a) + \frac{1}{2} [W(0) + W_r(2a)] L_i^2(a) = \\ & = \frac{1}{2} W(0) |L^2(a)| - \frac{1}{2} [W_r(2a) L_r^2(a) + 2W_i(2a) L_r(a) L_i(a) - W_r(2a) L_i^2(a)] = \\ & = \frac{1}{2} W(0) |L^2(a)| - \frac{1}{2} \operatorname{Re}\{W^*(2a) L^2(a)\} . \end{aligned} \quad (15)$$

Coupling (14) and (15) together, the expression in (13) becomes

$$B(a) = 2 \frac{W(0) |L^2(a)| - \operatorname{Re}\{W^*(2a) L^2(a)\}}{W^2(0) - |W^2(2a)|} . \quad (16)$$

The required quantities here are available from (8) as

$$\begin{aligned} W(2a) &= \sum_k w_k \exp(-i2ak), \\ L(a) &= \sum_k w_k x_k \exp(-iak) . \end{aligned} \quad (17)$$

It is immediately obvious from (16) that  $B(a)$  can never be negative (presuming that the weights are nonnegative).

The general result for  $B(a)$  in (16) is the quantity that must be maximized by choice of frequency  $a$ . However, it is interesting to observe that for frequencies where the window is small, that is,

$$|W(2a)| \ll W(0) , \quad (18)$$

then (16) simplifies to

$$B(a) \approx \frac{2}{W(0)} |L^2(a)| = \frac{2}{W(0)} \left| \sum_k w_k x_k \exp(-iak) \right|^2 , \quad (19)$$

which is identical to (4). Thus, for those frequencies where (18) is true, the function  $B(a)$  is approximately the magnitude-squared Fourier transform of the weighted data; this corresponds to values of  $a$  not near multiples of  $\pi$ .

#### PROPERTIES OF $B(a)$

Since  $W(2a)$  has period  $\pi$  in  $a$ , while  $L(a)$  has period  $2\pi$  in  $a$ , the function  $B(a)$  in (16) must have period  $2\pi$  in  $a$ ; that is,

$$B(a + 2\pi) = B(a) . \quad (20)$$

But at the same time, we have even property

$$B(-a) = B(a) , \quad (21)$$

because  $L(-a) = L^*(a)$ ,  $W(-2a) = W^*(2a)$ , using the realness of sequences  $\{w_k\}$  and  $\{x_k\}$ . What this means is that we only need to

$$\text{compute } B(a) \text{ for } 0 \leq a \leq \pi , \quad (22)$$

since all other values can be obtained therefrom. Reference to (6) reveals that  $\omega$  is being varied over the range  $(0, \pi/\Delta)$ , or that cyclic frequency  $f = \omega/(2\pi)$  is varying over  $(0, .5/\Delta)$ . This latter range extends up to the Nyquist frequency, as expected.

VALUE OF  $B(a)$  AS  $a \rightarrow 0$

If we substitute  $a = 0$  in (16), we get  $B(0) = 0/0$ , which is indeterminate. Hence, for small  $a$ , we expand (17) according to

$$\begin{aligned} W(2a) &\sim W_0 - i2aW_1 - 2a^2W_2, \\ L(a) &\sim L_0 - iaL_1 - \frac{1}{2}a^2L_2, \end{aligned} \quad (23)$$

where  $n$ -th order "moments"

$$\begin{aligned} W_n &= \sum_k w_k k^n, \\ L_n &= \sum_k w_k x_k k^n. \end{aligned} \quad (24)$$

Substitution of (23) in (16) and simplification yields

$$\lim_{a \rightarrow 0} B(a) = \frac{W_0L_1^2 - 2W_1L_1L_0 + W_2L_0^2}{W_0W_2 - W_1^2}. \quad (25)$$

This limiting result is the same value that is attained as if we minimized weighted error

$$E = \sum_k w_k [x_k - \mu - vk]^2, \quad (26)$$

by choice of constant value  $\mu$  and linear trend  $vk$ . In fact, direct minimization of (26) yields optimum coefficients

$$\mu_0 = \frac{W_2L_0 - W_1L_1}{W_0W_2 - W_1^2}, \quad v_0 = \frac{W_0L_1 - W_1L_0}{W_0W_2 - W_1^2}, \quad (27)$$

and associated minimum error

$$E_0 = \sum_k w_k x_k^2 - \frac{w_0 L_1^2 - 2w_1 L_1 L_0 + w_2 L_0^2}{w_0 w_2 - w_1^2}. \quad (28)$$

As claimed above, the last term in (28) is precisely the result given by (25); see (12) also. Thus, the limit, as  $a \rightarrow 0$ , of model fit (7) is the best-fitting constant plus linear trend to the given data. This can be obtained from (7) only if quadrature coefficient  $\beta$  behaves as  $1/a$  as  $a \rightarrow 0$ . Indeed, in a later section, we will show that this is precisely the behavior of  $\beta$  in this limit. Thus, setting  $a = 0$  in (7) and keeping  $\beta$  finite does not lead to the result in (25) and (28), but instead gives only the best fitting constant. We will allow the more general fit afforded by (26) here, and will utilize the value achieved by (25) in the limit, as  $a \rightarrow 0$ .

VALUE OF  $B(a)$  AS  $a \rightarrow \pi$

If we substitute  $a = \pi$  in (16), there follows  $B(\pi) = 0/0$ , which is indeterminate. However, if we let  $a = \pi + \tilde{a}$ , we see from (17) that

$$\begin{aligned} W(2a) &= W(2\pi + 2\tilde{a}) = W(2\tilde{a}), \\ L(a) &= L(\pi + \tilde{a}) = \sum_k w_k (-1)^k x_k \exp(-i\tilde{a}k). \end{aligned} \quad (29)$$

Thus,  $W(2\tilde{a})$  behaves the same about  $\tilde{a} = 0$  as  $W(2a)$  does about  $a = 0$ . Also, the last term in (29) behaves the same about  $\tilde{a} = 0$  as  $L(a)$  does about  $a = 0$ , provided that each data element  $x_k$  is replaced by  $(-1)^k x_k$ . Thus, (25) can be immediately utilized to yield the result

$$\lim_{a \rightarrow \pi} B(a) = \frac{w_0 \tilde{L}_1^2 - 2w_1 \tilde{L}_1 \tilde{L}_0 + w_2 \tilde{L}_0^2}{w_0 w_2 - w_1^2}, \quad (30)$$

where moments (24) have been replaced by

$$\tilde{L}_n = \sum_k (-1)^k w_k x_k k^n. \quad (31)$$

Physical interpretation of result (30) is similar to that given earlier for  $a \gg 0$  in (25)-(28). Namely, in the limit as  $a \gg \pi$ , the best constant plus linear trend is fitted to alternating data  $\{(-1)^k x_k\}$ . Again, this requires quadrature coefficient  $\beta$  in fit (7) to behave like  $1/(a - \pi)$  as  $a \gg \pi$ .

#### EXAMPLE OF EQUAL WEIGHTS

Let weights

$$w_k = \frac{1}{N} \quad \text{for} \quad 1 \leq k \leq N. \quad (32)$$

Then window (17) becomes

$$W(2a) = \frac{\sin(Na)}{N \sin(a)} \exp(-i(N+1)a). \quad (33)$$

This is the example considered in [9].

The moments (24) for this case are given by

$$\begin{aligned} w_0 &= 1, \quad w_1 = \frac{1}{2}(N+1), \quad w_2 = \frac{1}{6}(N+1)(2N+1), \\ w_0 w_2 - w_1^2 &= \frac{1}{12}(N^2 - 1). \end{aligned} \quad (34)$$

The numerator of (25) is then

$$\begin{aligned} L_1^2 - (N+1)L_1 L_0 + \frac{1}{6}(N+1)(2N+1)L_0^2 &= \\ &= \left(L_1 - \frac{N+1}{2} L_0\right)^2 + \frac{1}{12}(N^2 - 1)L_0^2 = \\ &= \frac{1}{N^2} \left[ \sum_k x_k \left(k - \frac{N+1}{2}\right) \right]^2 + \frac{N^2 - 1}{12N^2} \left[ \sum_k x_k \right]^2. \end{aligned} \quad (35)$$

Then (12), (25), and (34) yield

$$\lim_{a \rightarrow 0} E(a) = \frac{1}{N} \sum_k x_k^2 - \frac{1}{N^2} \left[ \sum_k x_k \right]^2 - \frac{12}{N^2(N^2 - 1)} \left[ \sum_k x_k \left( k - \frac{N+1}{2} \right) \right]^2, \quad (36)$$

which can be recognized as the minimum error for the best-fitting constant plus linear trend to data  $\{x_k\}$ .

## IN-PHASE AND QUADRATURE COEFFICIENTS

Since the modeling waveform in (7) is

$$\alpha \cos(ak) + \beta \sin(ak) = \operatorname{Re}\{(\alpha - i\beta) \exp(iak)\} , \quad (37)$$

the complex coefficient or strength of pure complex tone  $\exp(iak)$  is  $\alpha - i\beta$ . From (11) and (10), the numerator of  $\alpha_0 - i\beta_0$  is expressible as

$$\begin{aligned} A_{22}L_r(a) + A_{12}L_i(a) + iA_{11}L_i(a) + iA_{12}L_r(a) &= \\ &= \frac{1}{2}[W(0) - W_r(2a)]L_r(a) + i\frac{1}{2}[W(0) + W_r(2a)]L_i(a) + \\ &+ i\left(-\frac{1}{2}\right)W_i(2a)[L_r(a) - iL_i(a)] = \\ &= \frac{1}{2}W(0)L(a) - \frac{1}{2}W(2a)L^*(a) . \end{aligned} \quad (38)$$

Combining this with the denominator previously computed in (14), we have for the optimum complex coefficient,

$$\alpha_0 - i\beta_0 = 2 \frac{W(0)L(a) - W(2a)L^*(a)}{W^2(0) - |W^2(2a)|} . \quad (39)$$

For frequencies  $a$  such that the window is small relative to the origin value (see (18)), (39) simplifies to the approximate result

$$\alpha_0 - i\beta_0 \approx 2L(a) = 2 \sum_k w_k x_k \exp(-iak) , \quad (40)$$

which is just the Fourier transform of the weighted data.

If only the phase of the real tone is of interest, (39) indicates that

$$\arg(\alpha_0 - i\beta_0) = \arg(W(0)L(a) - W(2a)L^*(a)) . \quad (41)$$

If frequency  $a$  is known, this result is directly applicable; but if  $a$  is unknown, the value  $a$  that maximizes (16) must be used.

#### NORMALIZATION OF WEIGHTS

Without loss of generality, the sum of the weights  $\{w_k\}$  can be set equal to unity; that is, set

$$W(0) = \sum_k w_k = 1 . \quad (42)$$

Then the complex coefficient in (39) reduces to

$$\alpha_0 - i\beta_0 = 2 \frac{L(a) - W(2a)L^*(a)}{1 - |W^2(2a)|} , \quad (43)$$

while the maximizing function  $B(a)$  in (16) becomes

$$B(a) = 2 \frac{|L^2(a)| - \operatorname{Re}\{W^*(2a)L^2(a)\}}{1 - |W^2(2a)|} . \quad (44)$$

This slightly reduces the number of computations that have to be conducted and has been adopted in the program written here. This scaling is also retained in the following subsection.

#### INTERPRETATION OF (43)

An alternative form for coefficient (43) is

$$\alpha_0 - i\beta_0 = 2 \frac{L(a) - W(2a)L(-a)}{1 - |W^2(2a)|} , \quad (45)$$

where we utilized the realness of data  $\{x_k\}$  and weights  $\{w_k\}$ . This result can be interpreted as follows: the term

$$2L(a) = 2 \sum_k w_k x_k \exp(-iak) \quad (46)$$

is an estimate of the complex strength of the positive-frequency complex exponential  $\exp(iak)$  in the real data  $\{x_k\}$ , as modified by the weights. Similarly,  $2L(-a)$  estimates the strength of the term  $\exp(-iak)$  in the real data. The window  $W(2a)$  measures the amount of spillover from frequency  $-a$  to frequency  $a$ , that is, at separation  $2a$ , due to the weights  $\{w_k\}$ . This fraction (including phase information) of the spillover from negative frequencies to positive frequencies is subtracted from strength  $2L(a)$ . Finally, the denominator factor  $1 - |W^2(2a)|$  renormalizes the remainder according to the fractional spillover.

To justify this last scale factor, suppose that the data  $\{x_k\}$  contain a pure real tone at precisely the frequency  $a$ ; that is, let

$$\begin{aligned} x_k &= \alpha_0 \cos(ak) + \beta_0 \sin(ak) = \\ &= \operatorname{Re}\{(\alpha_0 - i\beta_0) \exp(iak)\} . \end{aligned} \quad (47)$$

Then (46) yields

$$\begin{aligned} 2L(a) &= \sum_k w_k [\alpha_0 (\exp(iak) + \exp(-iak)) - i\beta_0 (\exp(iak) - \exp(-iak))] \exp(-iak) = \\ &= \alpha_0 [1 + W(2a)] - i\beta_0 [1 - W(2a)] . \end{aligned} \quad (48)$$

Therefore,

$$2L(-a) = 2L^*(a) = \alpha_0 [1 + W^*(2a)] + i\beta_0 [1 - W^*(2a)] . \quad (49)$$

Therefore, the numerator of  $\alpha_0 - i\beta_0$  in (45) is

$$\begin{aligned}
& 2L(a) - W(2a)2L(-a) = \\
& = \alpha_0(1 + W) - i\beta_0(1 - W) - W[\alpha_0(1 + W^*) + i\beta_0(1 - W^*)] = \\
& = (\alpha_0 - i\beta_0)(1 - |W^2|) = (\alpha_0 - i\beta_0)[1 - |W^2(2a)|], \quad (50)
\end{aligned}$$

where we adopted the notational simplification  $W = W(2a)$  during the manipulations. Thus, the denominator factor  $1 - |W^2(2a)|$  in (45) is necessary to scale the amplitude back up to its correct value of  $\alpha_0 - i\beta_0$ .

#### VALUE OF COEFFICIENT AS $a \rightarrow 0$

We want to investigate the behavior of coefficient  $\alpha_0 - i\beta_0$  in (39) as  $a \rightarrow 0$ . (If we try to set  $a = 0$ , we get  $\alpha_0 - i\beta_0 = 0/0$ , which is indeterminate.) Accordingly, substitute expansions (23)–(24) into (39) and simplify to obtain the expression

$$\alpha_0 - i\beta_0 \sim \frac{W_2 L_0 - W_1 L_1}{W_0 W_2 - W_1^2} - \frac{i}{a} \frac{W_0 L_1 - W_1 L_0}{W_0 W_2 - W_1^2} \quad \text{as } a \rightarrow 0. \quad (51)$$

This result corroborates the claim made under (28) that  $\beta_0$  behaves as  $1/a$  as  $a \rightarrow 0$ . That is, the optimum quadrature coefficient of the pure real tone gets arbitrarily large as frequency  $a$  tends to zero.

If we combine (51) with the modelling function in (7), we have

$$\begin{aligned}
& \alpha_0 \cos(ak) + \beta_0 \sin(ak) \sim \alpha_0 + \beta_0 ak \sim \\
& \sim \frac{W_2 L_0 - W_1 L_1}{W_0 W_2 - W_1^2} + \frac{W_0 L_1 - W_1 L_0}{W_0 W_2 - W_1^2} k \quad \text{as } a \rightarrow 0, \quad (52)
\end{aligned}$$

which is precisely (26) and (27). Thus, the limit, as  $a \rightarrow 0$ , of modeling (7) is to fit the best constant plus linear trend to the data.

## FFT REALIZATION

For purposes of minimizing computations, we henceforth assume that the weights have been normalized according to (42); that is, their sum equals unity. This feature is incorporated in the following equations and the resultant program.

## MANIPULATION INTO FFT FORMS

According to (22), we are interested in evaluating  $B(a)$  in (44) over the range  $0 \leq a \leq \pi$ , where functions  $W$  and  $L$  are given by (17). Suppose then that we focus attention on values of frequency  $a$  given by

$$a = m \frac{2\pi}{M} \quad \text{for} \quad 0 \leq m \leq \frac{M}{2}. \quad (53)$$

Integer  $M$  will be chosen to be a power of 2, and is unrelated to  $N$ , the number of data points. Then (17) yields

$$L\left(m \frac{2\pi}{M}\right) = \sum_k w_k x_k \exp(-i2\pi mk/M) \equiv \mathcal{L}(m), \quad (54)$$

which is recognized as an  $M$ -size FFT of  $N$  nonzero real weighted data values  $\{w_k x_k\}$ .

At the same time, (17) also gives for the window

$$\begin{aligned} W\left(2m \frac{2\pi}{M}\right) &= \sum_k w_k \exp(-i2\pi 2mk/M) = \\ &= \sum_{j \text{ even}} w_{j/2} \exp(-i2\pi mj/M), \end{aligned} \quad (55)$$

where we let  $j = 2k$ . Now if we define sequence

$$d_j = \begin{cases} w_{j/2} & \text{for } j \text{ even} \\ 0 & \text{for } j \text{ odd} \end{cases}, \quad (56)$$

then (55) becomes

$$W\left(m \frac{2\pi}{M}\right) = \sum_j d_j \exp(-i2\pi mj/M) \equiv \mathcal{D}(m), \quad (57)$$

which is an  $M$ -size FFT of sequence  $\{d_j\}$ .

Direct employment of (54) and (57) in (44) yields

$$B\left(m \frac{2\pi}{M}\right) = 2 \frac{|\mathcal{L}^2(m)| - \operatorname{Re}\{\mathcal{D}^*(m) \mathcal{L}^2(m)\}}{1 - |\mathcal{D}^2(m)|}. \quad (58)$$

Thus, if we evaluate the two FFTs for  $\{\mathcal{L}(m)\}$  and  $\{\mathcal{D}(m)\}$  in (54) and (57), respectively, we have all the quantities necessary to determine  $B(m2\pi/M)$  for  $0 \leq m \leq M/2$ .

#### TWO REAL FFT'S VIA ONE COMPLEX FFT

Since (54) and (57) constitute FFTs of real sequences, they are not making full use of the capabilities of an FFT. To exploit the inherently complex nature of this tool, let

$$z_k = w_k x_k + i d_k \quad \text{for } 1 \leq k \leq 2N, \quad (59)$$

where sequence  $\{d_k\}$  was defined in (56). (Half of the real terms and half of the imaginary terms are zero in (59).) Then the FFT of size  $M$  of sequence (59) is

$$Z(m) = \sum_k z_k \exp(-i2\pi mk/M) = \mathcal{L}(m) + i \mathcal{D}(m), \quad (60)$$

where we presume that  $M > 2N$ . (Methods of circumventing this limitation are given in [10].)

Using the realness of sequences  $\{x_k\}$ ,  $\{w_k\}$ ,  $\{d_k\}$ , it follows from (54) and (57) that

$$Z^*(M - m) = \mathcal{L}^*(M - m) - i\mathcal{D}^*(M - m) = \mathcal{L}(m) - i\mathcal{D}(m) . \quad (61)$$

Now combining (60) and (61), we have

$$\begin{aligned} 2\mathcal{L}(m) &= Z(m) + Z^*(M - m) = S_x(m) + i D_y(m) , \\ 2\mathcal{D}(m) &= -i[Z(m) - Z^*(M - m)] = S_y(m) - i D_x(m) , \end{aligned} \quad (62)$$

where the real sum and difference functions are defined as

$$\begin{aligned} S_x(m) &= X(m) + X(M - m), \\ S_y(m) &= Y(m) + Y(M - m), \\ D_x(m) &= X(m) - X(M - m), \\ D_y(m) &= Y(m) - Y(M - m), \end{aligned} \quad (63)$$

in terms of the real and imaginary parts of FFT output  $Z(m)$  in (60), namely,

$$Z(m) = X(m) + i Y(m) . \quad (64)$$

Equation (62) accomplishes the decoupling of the FFT output  $Z(m)$  so as to yield the two desired FFTs  $\mathcal{L}(m)$  and  $\mathcal{D}(m)$  indicated in (54) and (57). However, it is advantageous to continue with the breakdown of these two complex sequences  $\mathcal{L}(m)$  and  $\mathcal{D}(m)$ , as done in (62), in terms of all the purely real quantities given in (63). For upon substitution of (62) in desired quantity  $B(m2\pi/M)$  in (58), we obtain the simplified form

$$B\left(m \frac{2\pi}{M}\right) = \frac{S_x^2(m)[2 - S_y(m)] + D_y^2(m)[2 + S_y(m)] + 2S_x(m)D_x(m)D_y(m)}{4 - (S_y^2(m) + D_x^2(m))} . \quad (65)$$

This latter form, which utilizes only real arithmetic, can be used only for  $0 < m < M/2$ . The values for  $B(0+)$  and  $B(\pi-)$  must come from (25) and (30), respectively, with  $w_0 = 1$ . A program for calculation of  $\{B(m2\pi/M)\}$  by means of (56), (59), (60), (63), and (65) is furnished in the appendix.

## SELECTION OF FFT SIZE M

It was presumed in (59) and (60) that FFT size  $M > 2N$ , where  $N$  is the number of data points, in order that there be an array element in location  $M-1$  available to receive data element  $id_{2N}$ . However, there is an additional reason for choosing  $M$  this large, having to do with the rate at which  $B(a)$  varies. The function  $B(a)$  in (44) depends critically on window function  $W(2a)$ . For equal weights, the results in (32) and (33) indicate that  $W(2a)$  changes significantly in an interval of length  $\pi/N$ ; in fact, this is the separation between zero crossings. In order to track this rapid variation in  $W$ , the increment  $2\pi/M$  in frequency  $a$  in (53) and (58) must be smaller than  $\pi/N$ . Thus, requirement  $M > 2N$  is a minimal requirement; in fact, it may be advantageous to consider  $M$  several times larger than  $N$ , if storage and FFT execution time are not excessive. Of course, the larger  $M$  is taken, the less fine-grain interpolation will be required later.

For other weightings than flat, such as Hanning, where the effective length of the weighting is foreshortened due to taper at the edges, the window function  $W$  is broader, and the condition on  $M$  is alleviated somewhat. However,  $M > 2N$  is a good rule of thumb to use in most cases.

## INTERPOLATION PROCEDURE

When the complete set of values of  $B(m2\pi/M)$  for  $0 \leq m \leq M/2$  are available, they are searched to find the maximum value. This maximum value and the two neighboring bin outputs ( $m$  values) are then used in a parabolic interpolation procedure to refine the estimate of the location of the best value of frequency  $a$  and the corresponding maximum value of  $B(a)$ .

Finally, this latter value of  $a$  can be used as a starting value for a fine-grained search, again by means of parabolic interpolation, in the neighborhood of this peak. These features are all incorporated in the accompanying program for this search procedure, where direct use of (44) is made; the previous FFT results are of no use in this final vernier estimation. Along with each estimated frequency  $a$ , the corresponding

coefficient  $\alpha_0 - i\beta_0$  is also estimated and printed out. A few stages of the vernier analysis suffice to give stable frequency estimates within the accuracy of the computer used here.

## RESULTS

An example of  $N = 25$  data points with FFT size  $M = 1024$  is displayed in figure 1, for the data sequence

$$x_k = \cos(k) + \frac{1}{2} \sin(k) \quad \text{for } 1 \leq k \leq N \quad (66)$$

and for the equal (or flat) weighting case of (32). The abscissa is normalized frequency  $a = \omega\Delta$ , and the ordinate is  $B(a)$  normalized relative to its peak value. The low-level sidelobes in figure 1 are due to the nature of the window  $W(2a)$ , given by (33) for this case.

The line labeled INITIAL gives the bin number  $J_s$  in which the peak is located. This bin and the two adjacent ones are then interpolated by means of a parabola to yield the initial value for  $B(a)$  labeled  $B_{ig}$  and the abscissa estimate  $a = 1.0000445$ . This value of  $a$  is then employed in subroutine SUB B to give the corresponding value  $B(a)$ .

In the next two lines of the print out, the above value of  $a$  is perturbed by  $\pm \Delta$ , the function  $B(a)$  is computed, and parabolic interpolation is again used on these three points to give the estimates labeled as REFINED values. Then this refined  $a$  value is used to recompute  $B(a)$  and indicated as the MAXIMUM value in the print out. Finally, the coefficient estimates  $\alpha$  and  $\beta$ , along with the minimum error,  $E_{\min}$ , are printed out.

The whole cycle of perturbation and parabolic interpolation is repeated in the next separated four lines of print out, but this time with  $\Delta$  decreased by a factor of 10. This cycle is repeated one final time in each of the figures presented. Prolonged repetition would result in excessive round-off error, due to the differencing of similar function values.

If the weighting is changed to Hanning,

$$w_k = 1 - \cos\left(\frac{2\pi k}{N+1}\right) \quad \text{for } 1 \leq k \leq N, \quad (67)$$

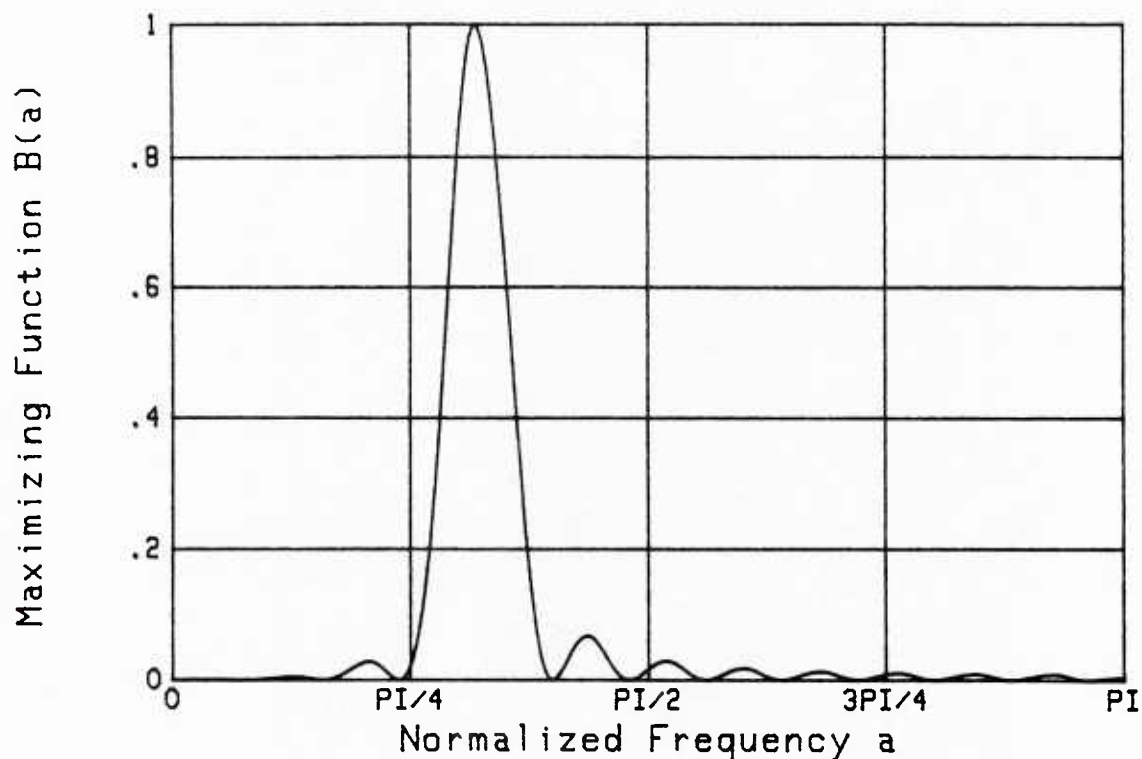
the corresponding results are displayed in figure 2. The initial estimate is  $a = .99999970$ , which is then refined to  $a = 1$ . The coefficients converge rapidly to the correct values, and the figure displays no visible sidelobes for this case of Hanning weighting. However, the window is broader.

The results of figures 3 and 4 correspond to figures 1 and 2, respectively, except that white noise of power  $1/12$  has been added to the waveform of (66). Now the Hanning weighting result in figure 4 also displays sidelobes, due to random fitting of the particular noise samples utilized. The refined values of  $a$  converge to  $a = .99318$  and  $a = 1.00208$ , respectively, which are not exactly correct, due to the additive noise. Also, the coefficient  $\alpha$  and  $\beta$  are considerably off their correct values, although the Hanning results in figure 4 are better than for the flat weighting used in figure 3.

Figures 5 and 6 are conducted for the two-tone data sequence

$$x_k = \cos(k) + \frac{1}{2} \sin(k) + \cos(2k) , \quad (68)$$

with no additive noise. The second peak near  $a = 2$  in these figures is due to the attempted match of model (7) to the data, when  $a$  is near 2. The program locks onto the stronger tone and indicates its frequency as  $a = .99447$  and  $a = 1.00128$ , respectively. Again, the estimates of frequency  $a$  and coefficients  $\alpha$  and  $\beta$  are better for the Hanning weighting in figure 6 than for the flat weighting in figure 5. This is due to the lower sidelobes of window  $W(2a)$  in the Hanning case.



```

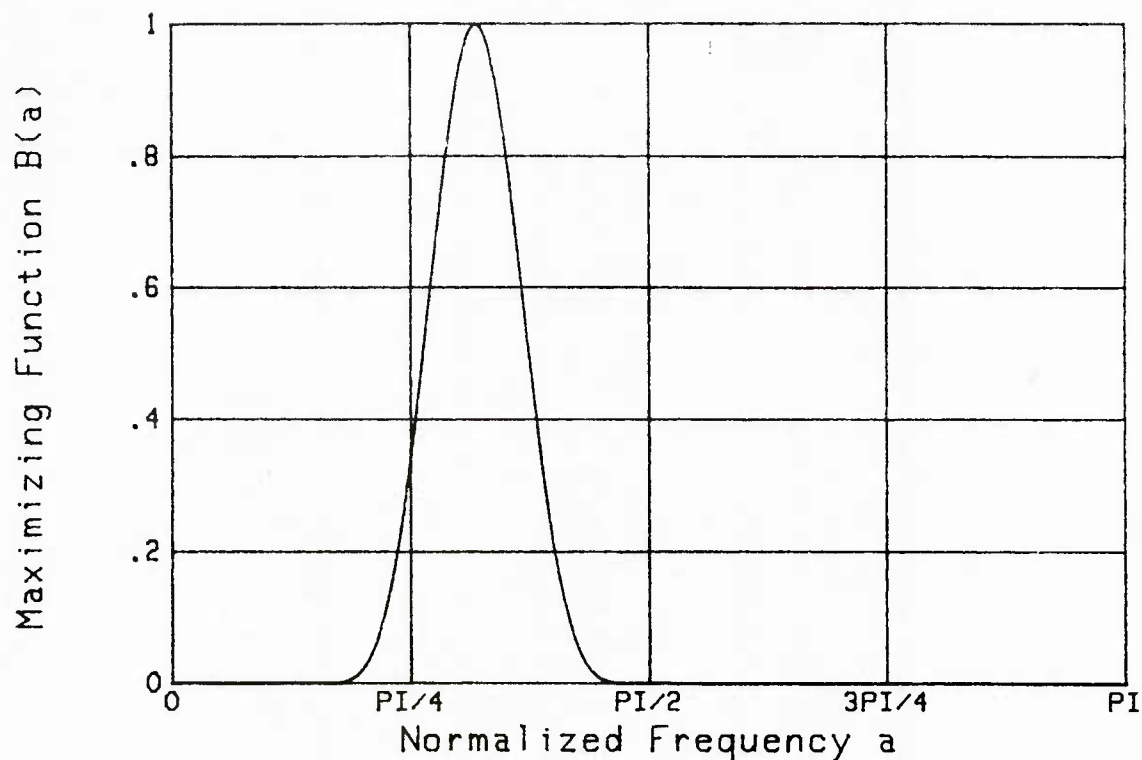
NUMBER OF DATA POINTS N = 25
SIZE OF FFT M = 1024
INITIAL: Js = 163   Big = .621074542628   a = 1.00004447034   B(a) = .62107486
REFINED a = 1.00000000426
REFINED B(a) = .621074930024
MAXIMUM B(a) = .621074930049
Alpha = .999999969685   Beta = .500000054303   Emin = 5.55111512313E-16

REFINED a = 1.00000000011
REFINED B(a) = .621074930049
MAXIMUM B(a) = .621074930049
Alpha = .99999999199   Beta = .500000001436   Emin = 1.11022302463E-16

REFINED a = 1
REFINED B(a) = .621074930049
MAXIMUM B(a) = .621074930049
Alpha = 1   Beta = .499999999994   Emin = 1.11022302463E-16

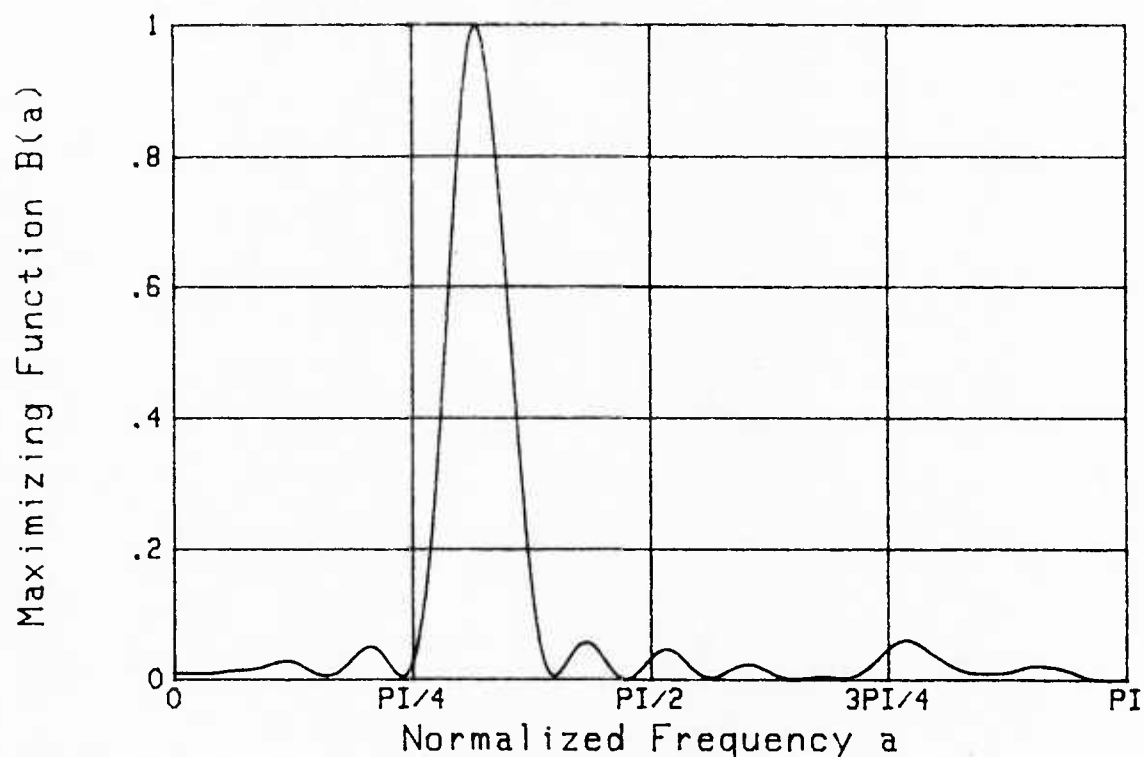
```

Figure 1. Flat Weighting, No Noise



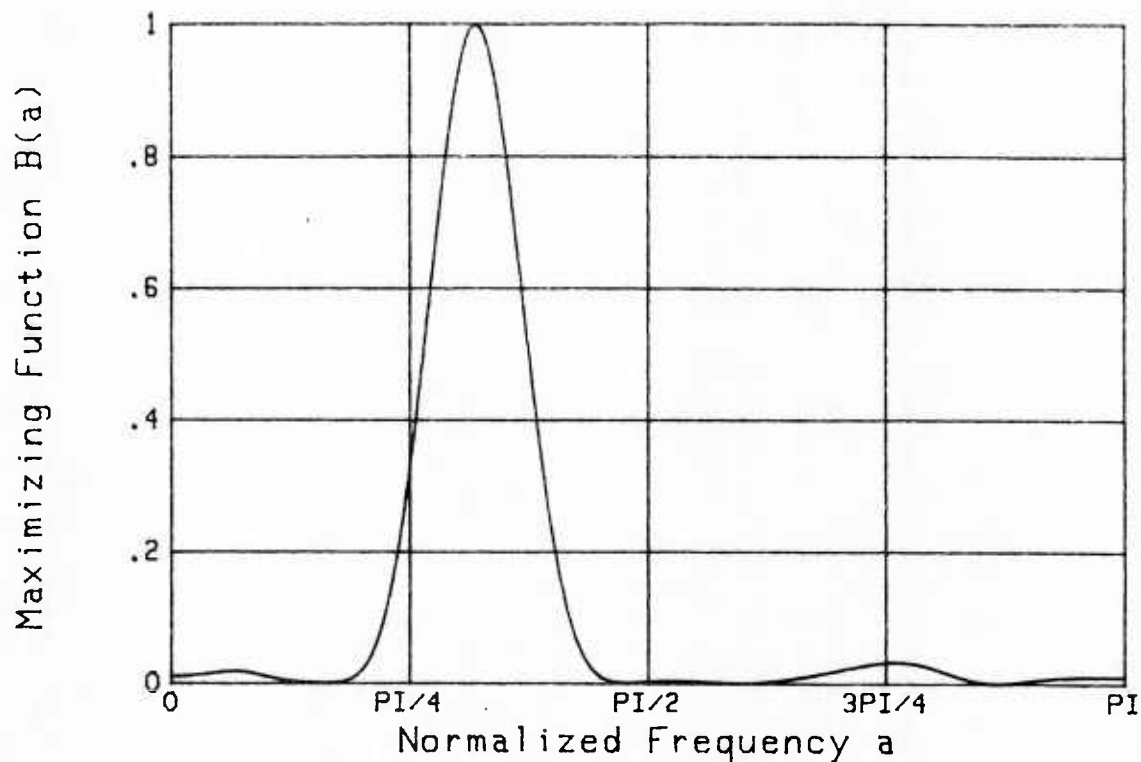
NUMBER OF DATA POINTS  $N = 25$   
 SIZE OF FFT  $M = 1024$   
 INITIAL:  $J_s = 163$   $Big = .624753874556$   $a = .999999700554$   $B(a) = .62475387$   
 REFINED  $a = .999999999908$   
 REFINED  $B(a) = .624753873395$   
 MAXIMUM  $B(a) = .624753873395$   
 Alpha =  $1.000000000059$  Beta =  $.499999999881$  Emin = 0  
  
 REFINED  $a = .999999999999$   
 REFINED  $B(a) = .624753873395$   
 MAXIMUM  $B(a) = .624753873395$   
 Alpha =  $1.000000000001$  Beta =  $.499999999986$  Emin =  $-1.11022302463E-16$   
  
 REFINED  $a = .999999999997$   
 REFINED  $B(a) = .624753873395$   
 MAXIMUM  $B(a) = .624753873395$   
 Alpha =  $1.000000000002$  Beta =  $.499999999996$  Emin =  $1.11022302463E-16$

Figure 2. Hanning Weighting, No Noise



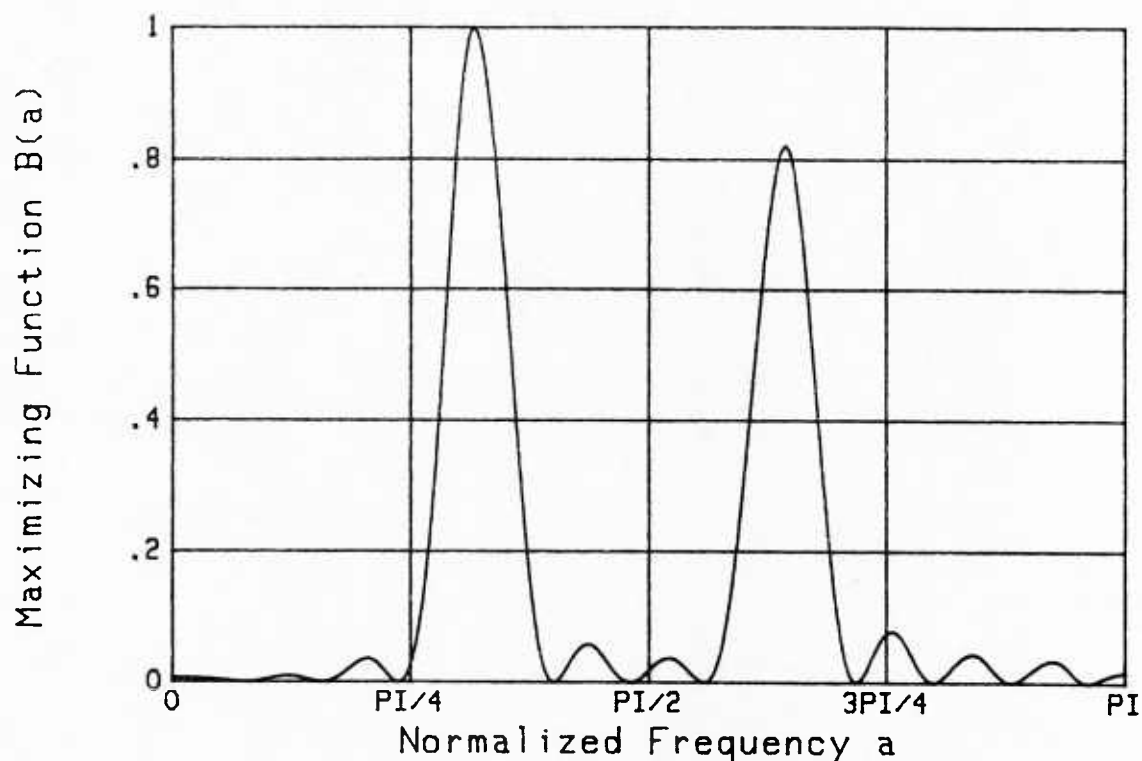
NUMBER OF DATA POINTS  $N = 25$   
 SIZE OF FFT  $M = 1024$   
 INITIAL:  $J_s = 162$   $B(a) = .657123186854$   $a = .993221368141$   $B(a) = .65712576$   
 REFINED  $a = .993178780656$   
 REFINED  $B(a) = .657125832748$   
 MAXIMUM  $B(a) = .657125832776$   
 Alpha = 1.1193876303      Beta = .282608428946      Emin = 6.39949448814E-02  
  
 REFINED  $a = .99317877595$   
 REFINED  $B(a) = .657125832776$   
 MAXIMUM  $B(a) = .657125832776$   
 Alpha = 1.11938765005      Beta = .282608363445      Emin = 6.39949448814E-02  
  
 REFINED  $a = .99317877584$   
 REFINED  $B(a) = .657125832776$   
 MAXIMUM  $B(a) = .657125832776$   
 Alpha = 1.11938765051      Beta = .282608361911      Emin = 6.39949448814E-02

Figure 3. Flat Weighting, Additive Noise



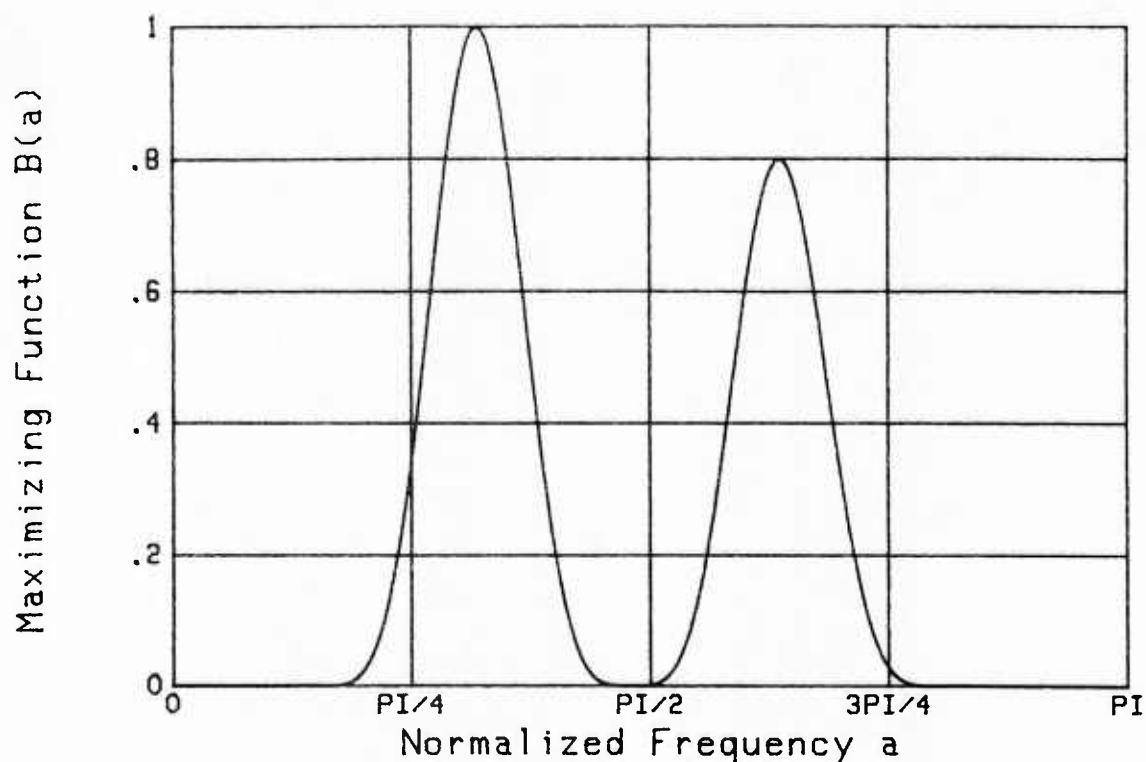
NUMBER OF DATA POINTS  $N = 25$   
 SIZE OF FFT  $M = 1024$   
 INITIAL:  $J_s = 163$   $Big = .643943182613$   $a = 1.00208436495$   $B(a) = .64394317$   
 REFINED  $a = 1.00208395996$   
 REFINED  $B(a) = .643943178345$   
 MAXIMUM  $B(a) = .643943178345$   
 Alpha = 1.0636516553      Beta = .396291769342      Emin = 5.15554013299E-02  
  
 REFINED  $a = 1.00208395965$   
 REFINED  $B(a) = .643943178345$   
 MAXIMUM  $B(a) = .643943178345$   
 Alpha = 1.06365165685      Beta = .396291765184      Emin = 5.15554013299E-02  
  
 REFINED  $a = 1.00208395966$   
 REFINED  $B(a) = .643943178345$   
 MAXIMUM  $B(a) = .643943178345$   
 Alpha = 1.06365165684      Beta = .396291765211      Emin = 5.15554013299E-02

Figure 4. Hanning Weighting, Additive Noise



NUMBER OF DATA POINTS  $N = 25$   
 SIZE OF FFT  $M = 1024$   
 INITIAL:  $J_s = 162$   $Big = .612307134756$   $a = .994519907011$   $B(a) = .6123056$   
 REFINED  $a = .994474013322$   
 REFINED  $B(a) = .612305684026$   
 MAXIMUM  $B(a) = .612305684054$   
 Alpha =  $1.03244095052$  Beta =  $.417726698818$  Emin =  $.499913742067$   
  
 REFINED  $a = .994474009397$   
 REFINED  $B(a) = .612305684054$   
 MAXIMUM  $B(a) = .612305684054$   
 Alpha =  $1.03244097595$  Beta =  $.417726642805$  Emin =  $.499913742067$   
  
 REFINED  $a = .994474009277$   
 REFINED  $B(a) = .612305684054$   
 MAXIMUM  $B(a) = .612305684054$   
 Alpha =  $1.03244097673$  Beta =  $.417726641086$  Emin =  $.499913742067$

Figure 5. Flat Weighting, Two Tones



```

NUMBER OF DATA POINTS N = 25
SIZE OF FFT M = 1024
INITIAL: Js = 163  Big = .623356825941  a = 1.00127460591  B(a) = .623356
REFINED a = 1.00127551942
REFINED B(a) = .623356862582
MAXIMUM B(a) = .623356862582
Alpha = .989782416069  Beta = .51724932123  Emin = .499950569458

REFINED a = 1.00127551958
REFINED B(a) = .623356862582
MAXIMUM B(a) = .623356862582
Alpha = .989782415052  Beta = .517249323177  Emin = .499950569458

REFINED a = 1.00127551958
REFINED B(a) = .623356862582
MAXIMUM B(a) = .623356862582
Alpha = .989782415038  Beta = .517249323204  Emin = .499950569458

```

Figure 6. Hanning Weighting, Two Tones

## SUMMARY

An automatic procedure for determining the best frequency, amplitude, and phase of a real tone fitted to discrete real data has been devised and programmed. It employs a single complex FFT for the initial search and then refines the estimates by simple parabolic interpolation procedures. The size  $M$  of the FFT is unrelated to the number  $N$  of data points, but should generally be taken at least equal to  $2N$  in order to guarantee adequate sampling in the frequency search. The user can input any real weights  $\{w_k\}$  of his choosing into the program; these are then automatically normalized to make their sum equal to unity.

The procedure is applicable to data records of any length  $N$ , without any approximations. However, if there is considerable noise in the data, then large  $N$  will be required in order to attain accurate estimates of the tone frequency, amplitude, and phase. This is not a drawback of the least squares procedure or program, but is a fundamental limitation of estimation capability in the presence of noise.

No derivatives of any of the error functions to be extremized are required in this approach. Instead, direct parabolic interpolation of the appropriate sampled functions is employed and can be carried through several stages to the desired degree of accuracy or until round-off error dominates. For  $a$  very near 0 or  $\pi$ , the approximation of  $B(a)$  by a parabola may not be adequate; special techniques may be required at these limits.

## APPENDIX

## PROGRAM FOR ESTIMATION OF TONE PARAMETERS

Inputs required of the user are in

line 10: N, number of data points,  
line 20: M, size of FFT.

The program is configured to accept up to  $N = 8000$  data points and an FFT size up to  $M = 16384$ . The user can also change the weighting from flat (in line 200) to whatever weighting is of interest. The appropriate window FFT is undertaken automatically, by means of lines 290 and 560. The initial estimate of  $a$  and the plot of  $B(a)$  are completed by line 1010. If refined estimates of  $a$  are desired, CONT EXECUTE must be performed, and can be repeated for additional refinement.

The terminology DOUBLE denotes INTEGER variables in BASIC on the HP 9000 computer. Subroutine SUB B computes  $B(a)$  and coefficients  $\alpha$  and  $\beta$  at any frequency  $a$  of interest. Generation of data for the examples here is accomplished in SUB Data, which must be replaced by the user to bring in his own data.

```

10      N=25                      !  NUMBER OF DATA POINTS
20      M=1024                    !  SIZE OF FFT; M > 2N REQUIRED
30      PRINT "NUMBER OF DATA POINTS N =";N
40      PRINT "SIZE OF FFT M =";M
50      DIM W(1:8000),Xd(1:8000),X(0:16383),Y(0:16383),Cos(0:4096)
60      REDIM W(1:N),Xd(1:N),X(0:M-1),Y(0:M-1),Cos(0:M/4)
70      DOUBLE N,M,M2,Ks,J2,M2    !  INTEGERS
80      IF M>2*N THEN 120
90      BEEP
100     PRINT "M <= 2N; INCREASE M OR DECREASE N."
110     PAUSE
120     T=2.*PI/M
130     FOR Ms=0 TO M/4
140     Cos(Ms)=COS(T*Ms)          !  QUARTER-COSINE TABLE
150     NEXT Ms
160     MAT X=(0.)
170     MAT Y=(0.)
180     S=0.
190     FOR Ks=1 TO N
200     Wk=1.                      !  SPECIFY WEIGHTS, k=1:N
210     W(Ks)=Wk
220     S=S+Wk
230     NEXT Ks
240     S=1./S
250     W1=W2=0.
260     FOR Ks=1 TO N
270     T=W(Ks)*S                  !  SCALE WEIGHTS
280     W(Ks)=T                   !  SO THAT SUM = 1
290     Y(Ks+Ks)=T
300     T=T*Ks
310     W1=W1+T                   !  MOMENTS OF
320     W2=W2+T*Ks                !  WEIGHTS
330     NEXT Ks
340     CALL Data(N,Xd(*))        !  FILL UP DATA ARRAY Xd(1:N)
350     So=To=Se=Te=En=0.
360     FOR Ks=1 TO N STEP 2
370     T1=Xd(Ks)
380     T2=W(Ks)*T1
390     X(Ks)=T2
400     So=So+T2
410     To=To+T2*Ks
420     En=En+T1*T2
430     NEXT Ks
440     FOR Ks=2 TO N STEP 2
450     T1=Xd(Ks)
460     T2=W(Ks)*T1
470     X(Ks)=T2
480     Se=Se+T2
490     Te=Te+T2*Ks
500     En=En+T1*T2              !  TOTAL WEIGHTED ENERGY
510     NEXT Ks

```

```

520   L0=Se+So                                !   MOMENTS
530   L1=Te+To                                !       OF
540   L0t=Se-So                               !   WEIGHTED
550   L1t=Te-To                               !       DATA
560   CALL Fft14(M,Cos(*),X(*),Y(*))
570   M2=M/2
580   T=W2-W1*M1                               !   EVALUATE B(a) IN X(0:M/2)
590   X(0)=(L1*L1-2.*W1*L1*L0+W2*L0*L0)/T
600   X(M2)=(L1t*L1t-2.*W1*L1t*L0t+W2*L0t*L0t)/T
610   FOR Ms=1 TO M2-1
620     T1=X(Ms)
630     T2=X(M-Ms)
640     Sx=T1+T2
650     Dx=T1-T2
660     T1=Y(Ms)
670     T2=Y(M-Ms)
680     Sy=T1+T2
690     Dy=T1-T2
700     T1=Dx*Sx*Dy
710     T2=4.-(Sy*Sy+Dx*Dx)
720     X(Ms)=(Sx*Sx*(2.-Sy)+Dy*Dy*(2.+Sy)+T1+T1)/T2
730   NEXT Ms
740   Big=X(0)                                !   SEARCH FOR MAXIMUM
750   Js=0
760   FOR Ms=1 TO M2
770     T=X(Ms)
780     IF T<=Big THEN 810
790     Big=T                                  !   MAXIMUM VALUE AND
800     Js=Ms                                  !   LOCATION IN ARRAY
810   NEXT Ms
820   IF Js>0 AND Js<M2 THEN 850
830   T=0.
840   GOTO 890
850   T1=X(Js+1)
860   T2=X(Js-1)
870   T=.5*(T1-T2)/(Big+Big-T1-T2) !   PARABOLIC INTERPOLATION
880   Big=Big+.25*(T1-T2)*T           !   FOR MAXIMUM VALUE
890   As=(Js+T)*2.*PI/M               !   AND LOCATION OF MAXIMUM
900   CALL B(N,As,W(*),Xd(*),Alpha,Beta,Ba)
910   PRINT "INITIAL: ";Js="";Js;" Big=";Big;" a=";As;" B(a)=";Ba
920   GINIT
930   PLOTTER IS "GRAPHICS"
940   GRAPHICS ON
950   WINDOW 0.,M2,0.,Ba
960   GRID M2/8.,Ba/10.
970   FOR Ms=0 TO M2
980     PLOT Ms,X(Ms)                  !   PLOT B(a) OVER [0,PI]
990   NEXT Ms
1000  PENUP
1010  PAUSE
1020  GRAPHICS OFF

```

```

1030 Delta=1./M ! INITIAL SEARCH INCREMENT
1040 Delta=Delta*.1 ! FINE-GRAIN SEARCH
1050 CALL B(N,As-Delta,W(*),Xd(*),Alpha,Beta,Bam)
1060 CALL B(N,As+Delta,W(*),Xd(*),Alpha,Beta,Bap)
1070 T=.5*(Bap-Bam)/(Ba+Ba-Bam-Bap)
1080 IF ABS(T)<1. THEN 1100
1090 PRINT "REFINED INTERPOLATION IS BEYOND EDGES OF SEARCH INCREMENT: ";T
1100 As=As+T*Delta
1110 Ba=Ba+.25*(Bap-Bam)*T
1120 PRINT "REFINED a =";As
1130 PRINT "REFINED B(a) =";Ba
1140 CALL B(N,As,W(*),Xd(*),Alpha,Beta,Ba)
1150 PRINT "MAXIMUM B(a) =";Ba
1160 Emin=En-Ba ! MINIMUM ENERGY
1170 PRINT "Alpha = ";Alpha;" Beta = ";Beta;" Emin =";Emin
1180 PRINT
1190 PAUSE
1200 GOTO 1040
1210 END
1220 !
1230 SUB B(DOUBLE N,REAL As,W(*),Xd(*),Alpha,Beta,Ba)
1240 DOUBLE Ks
1250 A2=As+As
1260 Wr=Wi=Lr=Li=0.
1270 FOR Ks=1 TO N
1280 Tw=W(Ks)
1290 Tx=Tw*Xd(Ks)
1300 T1=As*Ks
1310 T2=A2*Ks
1320 Wr=Wr+Tw*COS(T2)
1330 Wi=Wi-Tw*SIN(T2)
1340 Lr=Lr+Tx*COS(T1)
1350 Li=Li-Tx*SIN(T1)
1360 NEXT Ks
1370 T1=(1.-Wr)*Lr
1380 T2=(1.+Wr)*Li
1390 T=2./(1.-(Wr*Wr+Wi*Wi))
1400 Alpha=(T1-Wi*Li)*T
1410 Beta=(Wi*Lr-T2)*T
1420 Ba=Wi*Lr*Li
1430 Ba=(T1*Lr+T2*Li-Ba-Ba)*T
1440 SUBEND
1450 !

```

```

1460 SUB Fft14(DOUBLE N,REAL Cos(*),X(*),Y(*)) ! N<=2^14=16384; 0 SUBS
1470 DOUBLE N1,N2,N3,N4,Log2n,J,K ! INTEGERS < 2^31 = 2,147,483,648
1480 DOUBLE I1,I2,I3,I4,I5,I6,I7,I8,I9,I10,I11,I12,I13,I14,L(0:13)
1490 IF N=1 THEN SUBEXIT
1500 IF N>2 THEN 1580
1510 A=X(0)+X(1)
1520 X(1)=X(0)-X(1)
1530 X(0)=A
1540 A=Y(0)+Y(1)
1550 Y(1)=Y(0)-Y(1)
1560 Y(0)=A
1570 SUBEXIT
1580 N1=N/4
1590 N2=N1+1
1600 N3=N2+1
1610 N4=N3+N1
1620 Log2n=1.4427*LOG(N)
1630 FOR I1=1 TO Log2n
1640 I2=2^(Log2n-I1)
1650 I3=I2+I2
1660 I4=N/I3
1670 FOR I5=1 TO I2
1680 I6=(I5-1)*I4+1
1690 IF I6<=N2 THEN 1730
1700 A1=-Cos(N4-I6-1)
1710 A2=-Cos(I6-N1-1)
1720 GOTO 1750
1730 A1=Cos(I6-1)
1740 A2=-Cos(N3-I6-1)
1750 FOR I7=0 TO N-I3 STEP I3
1760 I8=I7+I5-1
1770 I9=I8+I2
1780 T1=X(I8)
1790 T2=X(I9)
1800 T3=Y(I8)
1810 T4=Y(I9)
1820 A3=T1-T2
1830 A4=T3-T4
1840 X(I8)=T1+T2
1850 Y(I8)=T3+T4
1860 X(I9)=A1*A3-A2*A4
1870 Y(I9)=A1*A4+A2*A3
1880 NEXT I7
1890 NEXT I5
1900 NEXT I1

```

```

1910      I1=Log2n+1
1920      FOR I2=1 TO 14
1930          L(I2-1)=1
1940          IF I2>Log2n THEN 1960
1950          L(I2-1)=2^(I1-I2)
1960      NEXT I2
1970      K=0
1980      FOR I1=1 TO L(13)
1990          FOR I2=I1 TO L(12) STEP L(13)
2000              FOR I3=I2 TO L(11) STEP L(12)
2010                  FOR I4=I3 TO L(10) STEP L(11)
2020                      FOR I5=I4 TO L(9) STEP L(10)
2030                          FOR I6=I5 TO L(8) STEP L(9)
2040                              FOR I7=I6 TO L(7) STEP L(8)
2050                                  FOR I8=I7 TO L(6) STEP L(7)
2060                                      FOR I9=I8 TO L(5) STEP L(6)
2070                                          FOR I10=I9 TO L(4) STEP L(5)
2080                                              FOR I11=I10 TO L(3) STEP L(4)
2090                                                  FOR I12=I11 TO L(2) STEP L(3)
2100                                                      FOR I13=I12 TO L(1) STEP L(2)
2110                                                          FOR I14=I13 TO L(0) STEP L(1)
2120                                                              J=I14-1
2130                                                              IF K>J THEN 2200
2140                                                              A=X(K)
2150                                                              X(K)=X(J)
2160                                                              X(J)=A
2170                                                              A=Y(K)
2180                                                              Y(K)=Y(J)
2190                                                              Y(J)=A
2200                                                              K=K+1
2210                                                              NEXT I14
2220                                                              NEXT I13
2230                                                              NEXT I12
2240                                                              NEXT I11
2250                                                              NEXT I10
2260                                                              NEXT I9
2270                                                              NEXT I8
2280                                                              NEXT I7
2290                                                              NEXT I6
2300                                                              NEXT I5
2310                                                              NEXT I4
2320                                                              NEXT I3
2330                                                              NEXT I2
2340                                                              NEXT I1
2350              SUBEND
2360          !
2370          SUB Data(DOUBLE N,REAL Xd(*))
2380              DOUBLE Ks
2390              FOR Ks=1 TO N
2400                  Xd(Ks)=COS(Ks)+.5*SIN(Ks)
2410              NEXT Ks
2420          SUBEND

```

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